# Parameterized Complexity of Dynamic Belief Updates

Thomas Bolander[1] Arnaud Lequen[2]

DTU Compute, Technical University of Denmark

ENS Rennes, France

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# Outline

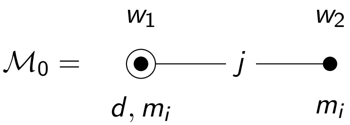
1. Dynamic Belief Update
2. Parameterized Complexity
3. New Complexity Results

# The coordinated attack problem in dynamic epistemic logic (DEL)

Two generals (agents), $i$ and $j$. They want to coordinate an attack, and only win if they attack simultaneously.

- $d$: "general $i$ will attack at dawn".
- $m_k$: the messenger is at general $k$ (for $k = i, j$).

Initial **epistemic model**:

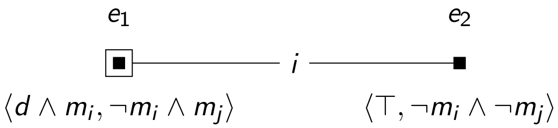



Nodes are **worlds**, edges are **indistinguishability edges** (S5 logic) (reflexive loops not drawn).

# The coordinated attack problem in dynamic epistemic logic (DEL)

Available **event models** $\mathcal{E}_{i \to j}$ (send message $d$ from $i$ to $j$) and $\mathcal{E}_{j \to i}$ (send message $d$ from $j$ to $i$).

$\mathcal{E}_{i \to j} =$

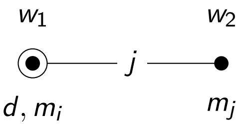



And symmetrically for $\mathcal{E}_{j \to i}$ (recall $d$: $i$ attacks at dawn; $m_k$: messenger is at general $k$).

Nodes are **events**, and each event is labelled by $\langle pre, post \rangle$ where *pre* is a **precondition** (epistemic formula) and *post* is a **postconditions** (conjunction of literals) [Baltag et al., 1998; van Ditmarsch et al., 2006; Bolander et al., 2011].

# The product update in DEL

$\mathcal{M}_0 =$

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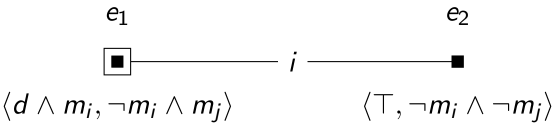


$\mathcal{M}_0 \otimes \mathcal{E}_{i \to j} =$

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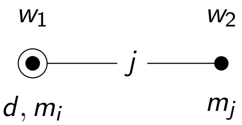
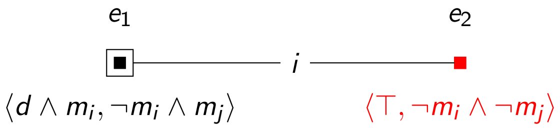



$\mathcal{E}_{i \to j} =$

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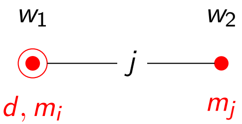


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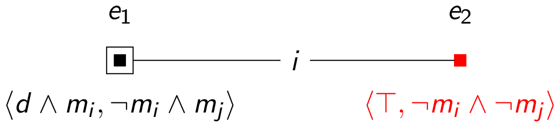


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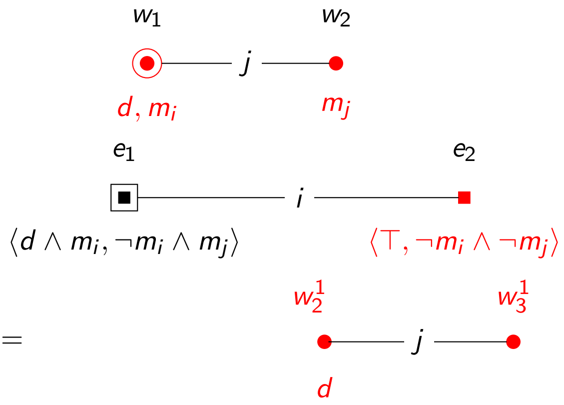

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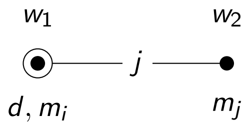


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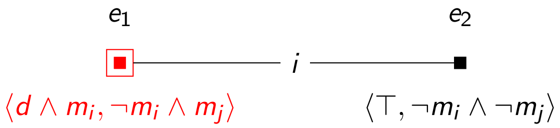


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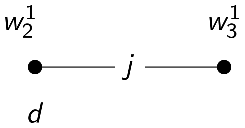

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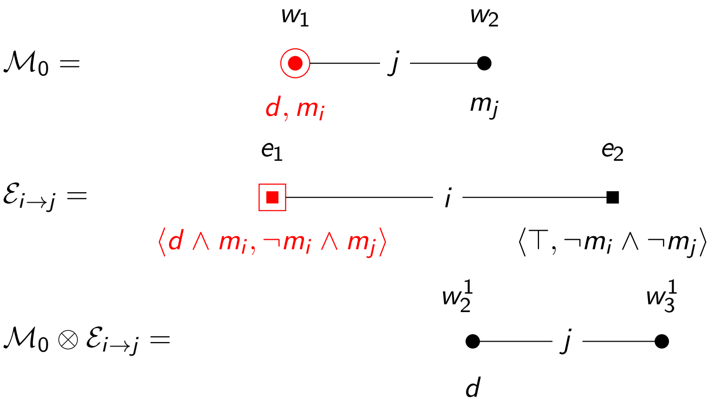

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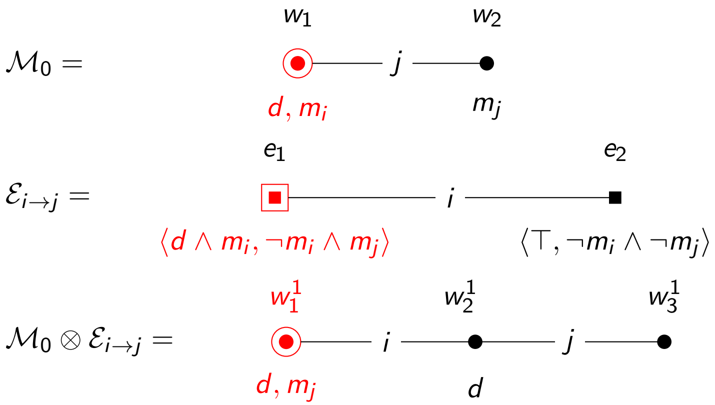

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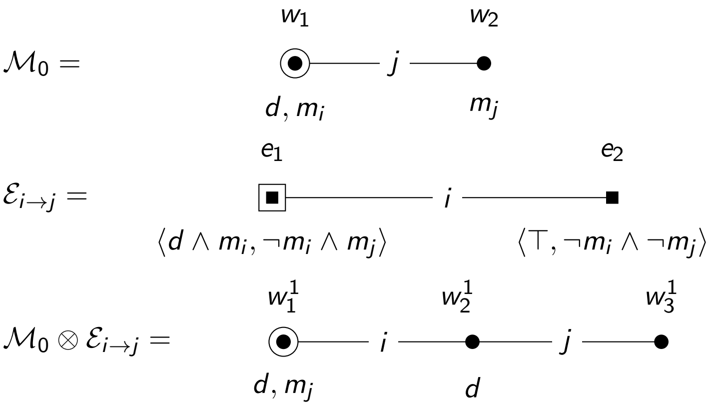

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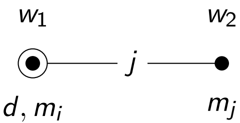


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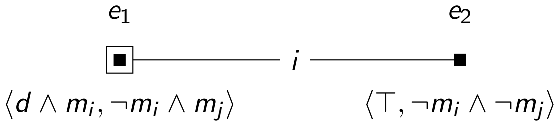
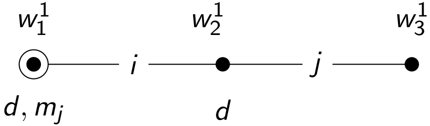
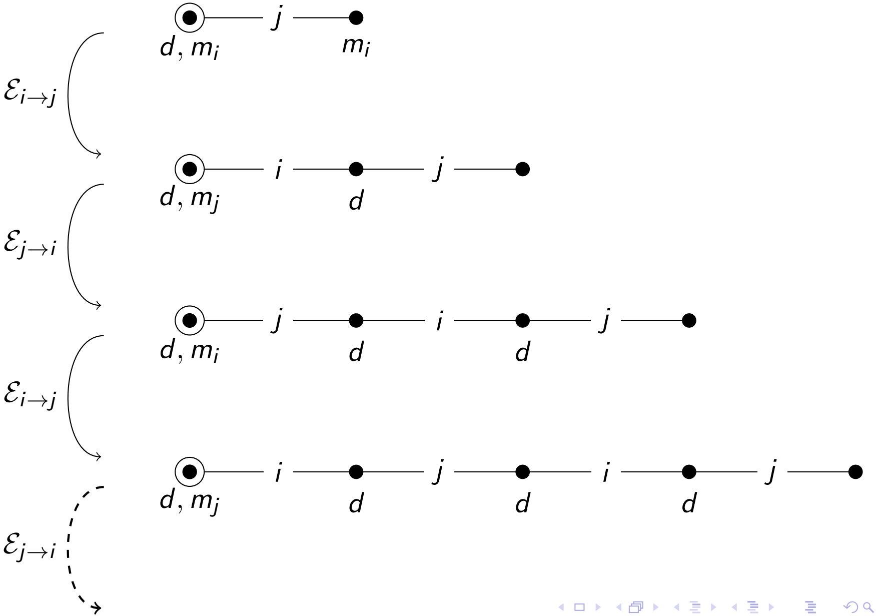



$\mathcal{M}_0 \otimes \mathcal{E}_{i \to j} =$




- $\mathcal{M}_0 \models \neg K_j d$
- $\mathcal{M}_0 \otimes \mathcal{E}_{i \to j} \models K_i d \wedge K_j d \wedge \neg K_i K_j d$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$\mathcal{E}_{j \to i}$

# The Dynamic Belief Update (DBU) Problem

**Dynamic Belief Update (DBU) [van de Pol et al., 2018]**

- **Input:** An epistemic model $\mathcal{M}$, a series of event models $\mathcal{E}_1, \ldots, \mathcal{E}_u$ and an epistemic goal formula $\varphi_g$
- **Output:** *Yes* if $\mathcal{M} \otimes \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_u \models \varphi_g$. Otherwise *No*.

DBU can also be seen as the plan verification problem in epistemic planning.

**Problem: Complexity [van de Pol et al., 2018]**

- DBU is intractable (PSPACE-complete)

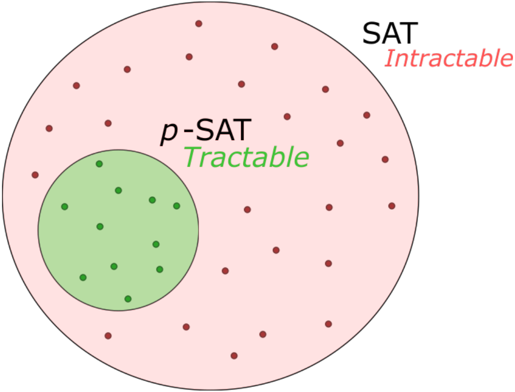
# History

## Theory of Mind reasoning

- The **cognitive capacity** called *Theory of Mind* (*ToM*) is our ability to attribute mental states to oneself and to others.
- Applications in psychology, philosophy, cognitive neuroscience, etc.

- DBU introduced by van de Pol et al. [2018] to study ToM
- Motivation: Understand which aspects are responsible for the intractability of ToM reasoning

# Parameterized Complexity




## Parameterized variant

p-SAT: variant of SAT where the number of variables is constant, equal to p

# Ins and outs of DBU

**Idea: Parameterized Complexity**

Bound some dimensions of the problem: the number of agents, the maximum size of an event model, etc.

Hardness results for DBU immediately give hardness results for:

- Plan existence in epistemic planning
- DEL model checking

# In this presentation

## Approach

- Study the tractability of a parameterized version of DBU, and its $2^7 = 128$ sub-problems, of which 96 are unique.

## Our contribution

- 8 out of 10 problems left open by van de Pol et al. [2018] now have their decidability settled
- Alternative proofs for 84 decidability results

(We implemented a small tool to keep track of all problems and their interdependencies, including which problems are still open.)

# Outline

# Parameters for DBU

## Parameters

- **Idea:** put a bound on some dimensions of the problem
- Problems of the form $X$-DBU where $X$ is a set of parameters

| Parameter | Description |
|---|---|
| $a$ | # of agents |
| $c$ | Max size of event precondition |
| $e$ | Max # of events in an event model |
| $f$ | Size of goal formula $\varphi_g$ |
| $o$ | Model depth of goal formula $\varphi_g$ |
| $p$ | # of propositional variables |
| $u$ | # of event models |

# Fixed-Parameter Tractability

## Fixed-parameter tractable problems

A problem is *fixed-parameter tractable* (*FPT*) if there is an algorithm that decides every instance $\omega$ with parameters $k_1, \ldots, k_n \in \mathbb{N}$ in time

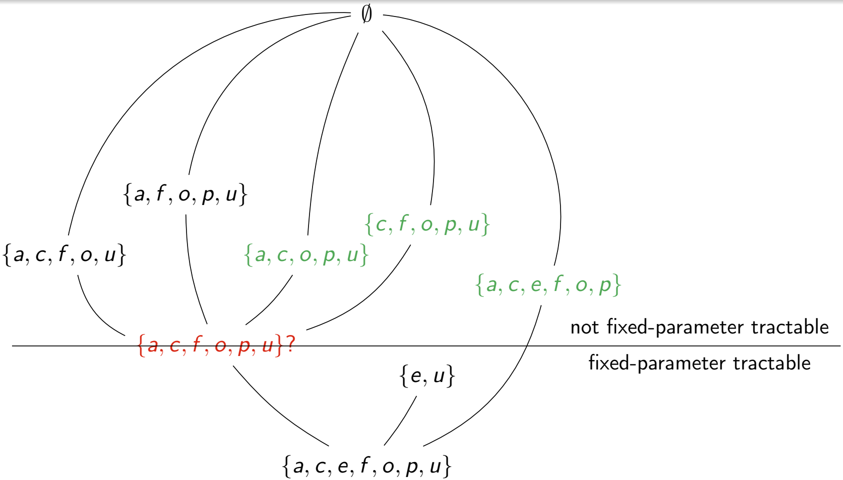
$$f(k_1, \ldots, k_n) \cdot P(|\omega|)$$

- $f$ is a computable function
- $P$ is a polynomial

## Tractability of $\{e, u\}$-DBU [van de Pol et al., 2018]

- $\{e, u\}$-DBU is FPT, i.e., is tractable when the number of event models ($u$) and number of events per model ($e$) are fixed
- There exists an algorithm for $\{e, u\}$-DBU in $\mathcal{O}(e^u \cdot P(|\omega|))$

# Current tractability results for DBU




- In black: results of van de Pol et al. [2018]
- In green: our results

# Outline

1. Dynamic Belief Update
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3. New Complexity Results
   - {a, c, e, f, o, p}-DBU
   - {c, f, o, p, u}-DBU
   - {a, c, o, p, u}-DBU

# $\{a, c, e, f, o, p\}$-DBU

## Result

$\{a, c, e, f, o, p\}$-DBU is intractable, that is, if we bound the number of propositional variables and agents $(a, p)$, the size of the preconditions of events and the number of events $(c, e)$, and the length and modal depth of the goal formula $(f, o)$, then DBU remains intractable.

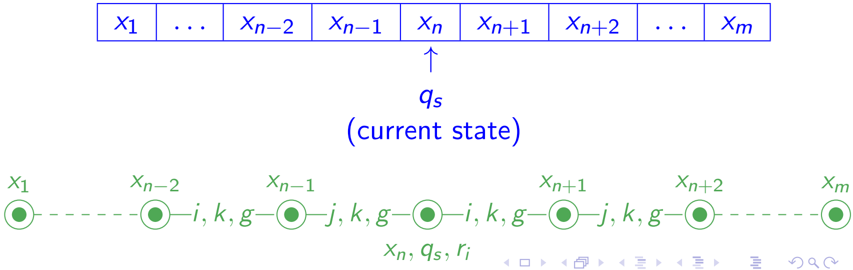
## Idea

- Encode a *fixed* non-deterministic Turing machine $M$, that solves an NP-complete problem in polynomial time, into an instance of $\{a, c, e, f, o, p\}$-DBU

# Proof - Sketch

**Idea:** *Adapted from Bolander et al. [2011]*

- Encode a non-deterministic Turing machine into an instance of DBU
- Done by representing each reachable configuration of $M$ by an epistemic model encoding the tape and state: an alternating chain of worlds labelled by state and tape symbols.



# Proof - Conclusion

- Introduce the event model $\mathcal{E}_{step}$ that simulates one non-deterministic step of $M$
- Then $\mathcal{M} \otimes (\mathcal{E}_{step})^n$ is the epistemic model that encodes all states of $M$ reachable by applying $n$ transitions (computation steps) to the state encoded in $\mathcal{M}$

**Max runtime for $M$ solving NP-complete problem**

There exists a polynomial $P$ such that there is a run where $M$ accepts a positive instance $\omega$ in time $P(|\omega|)$

- $M$ accepts $\omega$ iff $\mathcal{M}_0 \otimes (\mathcal{E}_{step})^{P(|\omega|)} \models \hat{K}_g q_f$ (where $q_f$ is the accepting state)

**Conclusion**

Fpt-reduction from an NP-complete prob. to $\{a, c, e, f, o, p\}$-DBU

# $\{c, f, o, p, u\}$-DBU

## Result

$\{c, f, o, p, u\}$-DBU is intractable, that is, if we bound the number of propositional variables ($p$), the size of preconditions of events and the number of event updates ($c, u$), and the size and model depth of the goal formula ($f, o$), then DBU remains intractable.

## Reduction

Fpt-reduction from k-W2SAT, the problem of deciding whether a 2CNF propositional formula $\psi$ can be satisfied with at most k variables set to true. Reduction inspired by van de Pol [2018].

## Ideas

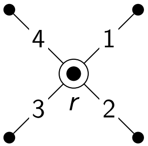
- Epistemic models encode a set of valuations via *valuation gadgets*
- Event models multiply and modify these valuation gadgets

# $\{c, f, o, p, u\}$-DBU - Sketch

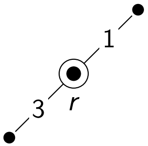
## Valuation gadget

A *valuation gadget* $\mathcal{M}_v$ encodes the valuation $v$ into an epistemic (sub)model. $\mathcal{M}_v$ has a root (designated world) marked by the propositional variable $r$, and an outgoing $i$-edge iff $x_i$ if true $v$.

- $P = \{x_1, \ldots, x_4\}$




$$\left\{ \begin{array}{l} x_1 \mapsto \bot,\ x_2 \mapsto \bot \\ x_3 \mapsto \bot,\ x_4 \mapsto \bot \end{array} \right\}$$

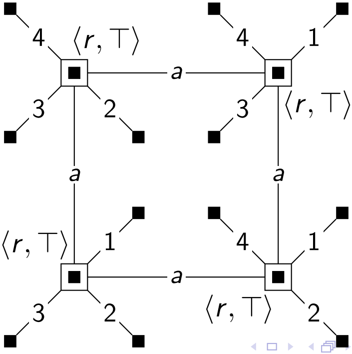



$$\left\{ \begin{array}{l} x_1 \mapsto \bot,\ x_2 \mapsto \top \\ x_3 \mapsto \bot,\ x_4 \mapsto \top \end{array} \right\}$$

# $\{c, f, o, p, u\}$-DBU - Sketch

## Event model

Event model $\mathcal{E}_{setOneTrue}$ constructs, for each submodel $\mathcal{M}_v$, $m$ new gadgets $\mathcal{M}_{v[x_1 \mapsto \top]}, \ldots, \mathcal{M}_{v[x_m \mapsto \top]}$, where $m$ is the number of variables of $\psi$



# $\{c, f, o, p, u\}$-DBU - Sketch

## Checking model

- Our proof is inspired by the proof of intractability of $\{c, o, p, u\}$-DBU by van de Pol et al. [2018].
- **Idea:** Introduce an event model named $\mathcal{E}_{check}$ that compresses the final epistemic model, so that our desired property can be checked with a fixed size goal formula.

- Initial epistemic model $\mathcal{M}_\perp$ encodes valuation $\{x_1 \mapsto \perp, \ldots, x_m \mapsto \perp\}$
- Goal formula: $\varphi_g := \hat{K}_a(r \wedge K_b \neg f)$
- $\psi$ is satisfiable with at most k variables set to true *iff* $\mathcal{M}_\perp \otimes (\mathcal{E}_{setOneTrue})^k \otimes \mathcal{E}_{check} \models \varphi_g$

# $\{a, c, o, p, u\}$-DBU

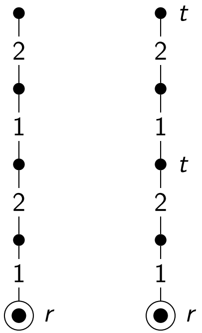
**Result**

$\{a, c, o, p, u\}$-DBU is intractable, that is, if we replace the previous bound on the size of the goal formula with a bound on the number of agents, then DBU remains intractable.

## Idea

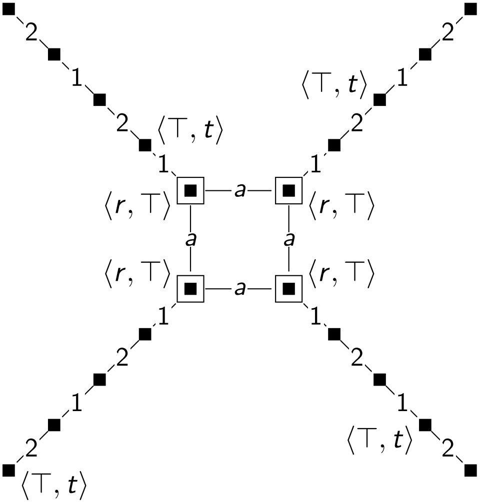
- Same overall idea as in previous proof
- Move the load from the number of agents to the size of the goal formula (parameter $f$ replaced by $a$)
- Main difference to previous proof: change the valuation gadgets to exploit depth of models rather than number of agents

# $\{a, c, o, p, u\}$-DBU - Sketch




$$\left\{ \begin{array}{l} x_1 \mapsto \bot \\ x_2 \mapsto \bot \\ x_3 \mapsto \bot \\ x_4 \mapsto \bot \end{array} \right\} \quad \left\{ \begin{array}{l} x_1 \mapsto \bot \\ x_2 \mapsto \top \\ x_3 \mapsto \bot \\ x_4 \mapsto \top \end{array} \right\}$$

Valuation gadgets



# Conclusion

## Solved problems

- From 10 open problems to 2 computationally equivalent ones

- Solved $\{a, c, p, e, f\}$-DBU by encoding a NTM into a DBU-instance. Gives many existing results as corollaries
- Solved $\{c, f, o, p, u\}$-DBU and $\{a, c, o, p, u\}$-DBU by generalising ideas from van de Pol et al. [2018]

## Future work

- Decide the complexity of $\{a, c, f, o, p, u\}$-DBU
- Consider additional relevant parameters
- Parameterized complexity of plan existence in epistemic planning